On the injectivity of assembly maps

Daniel Kasprowski

August 2019

Habilitationsschrift
zur Erlangung der venia legendi im Fach Mathematik

Mathematisch-Naturwissenschaftliche Fakultät der
Rheinischen Friedrich-Wilhelms-Universität Bonn

On the injectivity of assembly maps

Daniel Kasprowski

August 28, 2019

Contents

Zusammenfassung

Die algebraische K- und L-Theorie von Gruppenringen spielt eine wichtige Rolle in der Topologie. So lebt zum Beispiel Walls Endlichkeitshindernis in $\widetilde{K}_0$ vom integralen Gruppenring der Fundamentalgruppe und die Whiteheadtorsion, das Hindernis im s-Kobordismussatz, in einem Quotienten von K_1 vom integralen Gruppenring der Fundamentalgruppe.

Für jeden Ring R gibt es einen Funktor $\mathbf{K}_R$ von der Kategorie der transitiven G-Mengen $\operatorname{Or} G$ in die Kategorie der Spektren. Dieser schickt G/H auf ein zu $\mathbf{K}(R[H])$ äquivalentes Spektrum. Ein wichtiges Werkzeug zum Verständnis der K- und L-Theorie von Gruppenringen ist die Assembly-Abbildung

$$\operatorname*{hocolim}_{\operatorname{Or}_{\mathcal{F}} G} \mathbf{K}_R \to \mathbf{K}_R(*) \simeq \mathbf{K}(R[G]),$$

wobei $\mathcal{F}$ eine Familie von Untergruppen von G ist und $\operatorname{Or}_{\mathcal{F}} G$ die volle Unterkategorie von $\operatorname{Or} G$ der transitiven G-Mengen mit Stabilisator in $\mathcal{F}$. Das Hauptthema dieser Habilitationsschrift ist die Injektivität der Assembly-Abbildung. Auch wenn die meisten der vorgestellten Ergebnisse ein analoges Resultat in algebraischer L-Theorie haben, so werden wir hauptsächlich den K-theoretischen Fall betrachten.

Unter anderem werden Injektivitätssätze für lineare Gruppen und Untergruppen von zusammenhängenden Liegruppen behandelt, siehe [Kas15b, Kas16]. Diese sind ein Spezialfall von Gruppen mit endlicher Zerlegungskomplexität, eine Verallgemeinerung von endlicher asymptotischer Dimension. Endliche Zerlegungskomplexität wurde von Guentner, Tessera und Yu eingeführt [GTY13]. In Zusammenhang damit stehen [Kas17, KNR, Kas], die Teil dieser Habilitationsschrift sind. In diesen geht es hauptsächlich um Vererbungseigenschaften von endlicher Zerlegungskomplexität. Ein wichtiger Schritt im Beweis der Injektivität der Assembly-Abbildung ist die Konstruktion einer zugehörigen groben Homologietheorie. In [BEKWb] und weiteren Arbeiten mit Ulrich Bunke, Alexander Engel und Christoph Winges haben wir untersucht, inwieweit sich die Beweismethoden auf allgemeine grobe Homologietheorien verallgemeinern lassen.

1 Overview

The algebraic K-and L-theory of group rings plays an important role in topology. For example in Wall's finiteness obstruction or in the s-Cobordism Theorem due to Barden, Mazur and Stallings.

Theorem 1.1 (s-Cobordism Theorem)**.** *Let M_0 be a closed connected smooth manifold of dimension $n \geq 5$ with fundamental group π. Let $(W; M_0, f_0, M_1, f_1)$ be an h-cobordism, that is, the boundary inclusions $f_i\colon M_i \to W$ are homotopy equivalences. Then W is trivial over M_0, that is, diffeomorphic to $M_0 \times [0,1]$ relative to M_0, if and only if its Whitehead torsion $\tau(W, M_0)$, taking values in the Whitehead group $Wh(\pi) := K_1(\mathbb{Z}[\pi])/\{\pm g \mid g \in \pi\}$, vanishes.*

For every ring R there is a functor $\mathbf{K}_R\colon \operatorname{Or} G \to \mathbf{Sp}$ from the category of transitive G-sets and G-equivariant maps to the ∞-category of spectra, sending G/H to a spectrum equivalent to $\mathbf{K}(R[H])$. For a family of subgroups $\mathcal{F}$, the K-theoretic assembly map is the map

$$\operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}} G} \mathbf{K}_R \to \mathbf{K}_R(*) \simeq \mathbf{K}(R[G]),$$

where $\operatorname{Or}_{\mathcal{F}} G$ denotes the full subcategory of $\operatorname{Or} G$ of transitive G-sets with stabilizers in $\mathcal{F}$. The Farrell–Jones conjecture predicts that the assembly map (in K- and L-theory) for the family of virtually cyclic subgroups is an equivalence. It is by now known for many classes of groups, for example hyperbolic groups [BLR08], CAT(0)-groups [BL12, Weg12], solvable groups [Weg15] and mapping class groups [BB19]. The Farrell–Jones conjecture is an important tool in computing the K- and L-theory of group rings and for example implies the Borel conjecture about the topological rigidity of aspherical manifolds. If the Farrell–Jones conjecture holds for a group G, then the assembly map for the family of finite subgroups admits a left inverse, in particular it is split injective on homotopy groups. While this property is much weaker than the Farrell–Jones conjecture, it can be attacked by more elementary means and thus is known in broader generality. For example we have the following result about groups with finite decomposition complexity, which is a generalization of the concept of finite asymptotic dimension, see Section 5 for a definition. Although this and most other results mentioned in this thesis have an analogous version in L-theory, we will focus on the K-theoretic case from now on.

Theorem 1.2. *Let G be a group such that the family $\{H \backslash G\}_{H \in \mathcal{F}in}$ has finite decomposition complexity and assume that there is a finite dimensional G-CW-model for the classifying space for proper actions. Then for every ring R the assembly map* *for the family of finite subgroups*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}G} \mathbf{K}_R \to \mathbf{K}_R(*) \simeq \mathbf{K}(R[G])$$

admits a left inverse.

In Sections 2 to 6 this result and the necessary background are discussed. Sections 7 to 11 then contain an overview over the following articles which are part of this thesis.

[Kas15b]	*On the K-theory of subgroups of virtually connected Lie groups*, Algebr. Geom. Topol. 15 (2015), no. 6, 3467–3483
[Kas16]	*On the K-theory of linear groups*, Annals of K-Theory 1 (2016), no. 4, 441–456
[Kas17]	*The asymptotic dimension of quotients by finite groups*, Proc. Amer. Math. Soc. 145 (2017), no. 6, 2383–2389
[KW19]	*Algebraic K-theory of stable ∞-categories via binary complexes*, (joint with C. Winges), J. Topol. 12 (2019), no. 2, 442–462
[KNR]	*Regular finite decomposition complexity*, (joint with A. Nicas and D. Rosenthal), to appear in J. Topol. Anal.
[BEKWb]	*Injectivity results for coarse homology theories*, (joint with U. Bunke, A. Engel and C. Winges), arXiv:1809.11079
[Kas]	*Coarse embeddings into products of trees*, arXiv:1810.13361

The result of [Kas] is used to show that finite asymptotic dimension implies finite decomposition complexity and is discussed in Section 5. In [KNR] the notion of regular finite decomposition complexity is introduced. It is a strengthening of finite decomposition complexity sharing its inheritance properties that is also closed under forming quotients by finite groups. It is discussed in Section 7. In [Kas15b] and [Kas16] Theorem 1.2 is used to deduce injectivity results for linear groups and subgroups of virtually connected Lie groups. These results are discussed in Section 8. In [Kas17] it is proved that taking the quotient by an isometric action of a finite group does not change the asymptotic dimension. This is related to some open questions about the inheritance of FDC and finite asymptotic dimension to families of quotients by finite groups, see Section 9. [BEKWb] is part of a larger program to transfer the results for algebraic K-theory to general coarse homology theories. The statement of these results and the necessary background is given in Section 10. The article [KW19] shows that K-theory commutes with infinite products of stable ∞-categories. This is part of an ongoing project to generalize the previous results to K-theory with coefficients a stable ∞-category instead of an additive category, see Section 11.

2 Background on assembly maps

Instead of working with the K-theory of rings we will consider the K-theory of small additive G-categories. This generalizes the case of rings as the K-theory of a ring is equivalent to the K-theory of the category of finitely generated, projective modules over this ring. As before, for every small additive G-category $\mathbf{A}$ we have functor

$$\mathbf{K}_{\mathbf{A}} \colon \operatorname{Or} G \to \mathbf{Sp}$$

sending T to the spectrum $\mathbf{K}(\mathbf{A} *_G T)$, see [BR07a, Definition 3.1]. Here, the definition is such that $\mathbf{K}(\mathcal{P}r^R_{f.g.} *_G G/H) \simeq \mathbf{K}(R[H])$, where $\mathcal{P}r^R_{f.g.}$ is the category of finitely generated, projective R-modules. Note that here $\mathbf{K}$ always denotes non-connective algebraic K-theory, as for example defined by Schlichting [Sch06, Section 12].

Definition 2.1. Let G be a group. A *family of subgroups* of G is a non-empty set of subgroups of G that is closed under conjugation in G, and taking subgroups.

Example 2.2. The families of subgroups that we will mostly consider are the following.

- The family $\{1\}$ consisting only of the trivial subgroup;
- the family $\mathcal{F}in$ of finite subgroups;
- the family $\mathcal{V}Cyc$ of virtually cyclic subgroups and
- the family $\mathcal{A}ll$ of all subgroups.

Definition 2.3. For a family of subgroups $\mathcal{F}$ we define the *orbit category* $\operatorname{Or}_{\mathcal{F}} G$ as the full subcategory of $\operatorname{Or} G$ consisting of those transitive G-sets with stabilizers in the family $\mathcal{F}$.

Given two families of subgroups $\mathcal{F} \subseteq \mathcal{F}'$, we can form the *relative assembly map*

$$\alpha_{\mathcal{F}}^{\mathcal{F}'} \colon \operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}} G} \mathbf{K}_{\mathbf{A}} \to \operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}'} G} \mathbf{K}_{\mathbf{A}}.$$

In the case where $\mathcal{F}' = \mathcal{A}ll$, we have $\operatorname{colim}_{\operatorname{Or}_{\mathcal{A}ll} G} \mathbf{K}_{\mathbf{A}} \simeq \mathbf{K}_{\mathbf{A}}(*)$ and we call

$$\alpha_{\mathcal{F}} := \alpha_{\mathcal{F}}^{\mathcal{A}ll} \colon \operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}} G} \mathbf{K}_{\mathbf{A}} \to \mathbf{K}_{\mathbf{A}}(*)$$

the *assembly map for the family* $\mathcal{F}$. In the following $\mathbf{A}$ will always denote a small additive G-category.

For a family of subgroups $\mathcal{F}$ we define the *classifying space* $E_{\mathcal{F}}G$ to be a G-CW-complex X such that X only has stabilizers in $\mathcal{F}$ and for every $H \in \mathcal{F}$ we have $X^H \simeq *$. This property characterizes X up to G-homotopy equivalence. We call

$EG := E_{\{1\}}G$ the classifying space of G and $\underline{E}G := E_{\mathcal{F}in}G$ the classifying space for proper actions. For many of our results finiteness properties of $E_{\mathcal{F}}G$ play an important role, e.g. the existence of finite or finite-dimensional G-CW-models for $E_{\mathcal{F}}G$.

3 Geometric properties of groups

Our results will mostly rely on geometric properties of the involved groups. For this it is often useful to view a group as a metric space. The metric on a group is only well-defined up to coarse equivalence. Hence we will first review some basics of coarse geometry. Since it will be useful later, we will work with metric families instead of metric spaces.

A *metric family* is a set of metric spaces. A *map of metric families*, $F\colon \mathcal{X} \to \mathcal{Y}$, is a set of functions $f\colon X \to Y$, where $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, such that each element in $\mathcal{X}$ is the domain of at least one function in F.

The composition $G \circ F\colon \mathcal{X} \to \mathcal{Z}$ of $G\colon \mathcal{Y} \to \mathcal{Z}$ and $F\colon \mathcal{X} \to \mathcal{Y}$ is the set $\{g \circ f \mid f \in F, g \in G, \text{and the domain of } g \text{ is the range of } f\}$.

Definition 3.1. Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a map of metric families.

1. F is *coarse* (or *uniformly expansive*) if there exists a non-decreasing function

   $$\rho\colon [0,\infty) \to [0,\infty)$$

   such that for every $X \in \mathcal{X}, x, y \in X$, and $f\colon X \to Y$ in F,

   $$d_Y(f(x), f(y)) \leq \rho(d_X(x,y)).$$

   We call ρ the *control function for F*.

2. F is *effectively proper* if there exists a proper non-decreasing function

   $$\delta\colon [0,\infty) \to [0,\infty)$$

   such that for every $X \in \mathcal{X}, x, y \in X$, and $f\colon X \to Y$ in F,

   $$\delta(d_X(x,y)) \leq d_Y(f(x), f(y)).$$

3. F is a *coarse embedding* if it is both coarse and effectively proper.

4. F is *coarsely onto* if every $Y \in \mathcal{Y}$ is the range of some $f \in F$ and if there

exists a $C \geq 0$ such that for every $f\colon X \to Y$ in F and for every $y \in Y$ there exists an $x \in X$ such that $d_Y(f(x), y) \leq C$.

5. F is *close* to $F'\colon \mathcal{X} \to \mathcal{Y}$ if there exists a $C \geq 0$ with the property that for every $f\colon X \to Y$ in F (respectively, in F') there exists an $h\colon X \to Y$ in F' (respectively, in F) such that for all $x \in X$, $d_Y(f(x), h(x)) \leq C$.

6. F is a *coarse equivalence* if it is coarse and there exists a coarse map $G\colon \mathcal{Y} \to \mathcal{X}$ such that $G \circ F$ is close to the identity map of $\mathcal{X}$ and $F \circ G$ is close to the identity map of $\mathcal{Y}$.

A *subfamily* of a metric family $\mathcal{Y}$ is a metric family $\mathcal{U}$ such that every $U \in \mathcal{U}$ is a subspace of some $Y \in \mathcal{Y}$. The *inverse image* of $\mathcal{U}$ under the map $F\colon \mathcal{X} \to \mathcal{Y}$ is the subfamily of $\mathcal{X}$ given by $F^{-1}(\mathcal{U}) = \{f^{-1}(U) \mid U \in \mathcal{U}, f \in F\}$.

A metric family $\mathcal{X}$ is called *bounded* if $\sup_{X \in \mathcal{X}} \operatorname{diam} X < \infty$. The class of all bounded metric families is denoted by $\mathfrak{B}$.

Recall that a metric space X is the *r-disjoint union* of subspaces $\{X_i \mid i \in I\}$ if $X = \bigcup_{i \in I} X_i$, and for every $x \in X_i$ and $y \in X_j$ with $i \neq j$, $d(x, y) > r$. We denote an r-disjoint union by

$$X = \bigsqcup_{r\text{-disjoint}} \{X_i \mid i \in I\}.$$

Definition 3.2. A metric family $\mathcal{X}$ has *asymptotic dimension* at most n if for every $r > 0$ there exists $f(r) > 0$ such that the following holds. For every $X \in \mathcal{X}$ there is a decomposition $X = X_0 \cup X_1 \cup \cdots \cup X_n$ such that for each i, $0 \leq i \leq n$,

$$X_i = \bigsqcup_{r\text{-disjoint}} \{X_{ij} \mid j \in J_i\},$$

and $\operatorname{diam} X_{ij} \leq f(r)$ for all $0 \leq i \leq n,\ j \in J_i$. In particular, the metric family $\{X_{ij} \mid X \in \mathcal{X},\ 0 \leq i \leq n,\ j \in J_i\}$ is in $\mathfrak{B}$.

Any function $f\colon (0, \infty) \to \mathbb{R}$ with the above property is called a *control function* for X.

We say that a metric space X has asymptotic dimension at most n if the metric family $\{X\}$ consisting only of X has this property.

It is straightforward to see the following.

Lemma 3.3. *Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a coarse embedding and assume $\mathcal{Y}$ has asymptotic dimension at most n. Then $\mathcal{X}$ has asymptotic dimension at most n.*

Definition 3.4. Let G be a group. A *length function* on G is a function $l\colon G \to [0, \infty)$ satisfying

1. $l(g) = 0$ if and only if $g = 1$;
2. $l(g^{-1}) = l(g)$;
3. $l(gh) \leq l(g) + l(h)$.

A length function is *proper* if, for every $C \geq 0$, the set $\{g \in G \mid l(g) \leq C\}$ is finite.

One easily checks that a group admits a proper length function precisely when it is countable. If l is a length function on G,then $d(g,h) = l(g^{-1}h)$ defines a left-invariant metric on G. Conversely, by setting $l(g) = d(1,g)$ one checks that every left-invariant metric on G arises in this way. A length function is proper if and only if the corresponding metric is proper. In the literature, sometimes the possibility that some non-identity elements of G have length zero is allowed. In this case one only obtains a pseudometric.

It is straightforward to check that for any two proper, left-invariant metrics d, d' on G the identity induces a coarse equivalence

$$\mathrm{id}\colon (G,d) \to (G,d').$$

Hence we have the following proposition.

Proposition 3.5. *Every countable group admits a proper, left-invariant metric. Moreover, any two such metrics on a group are coarsely equivalent.*

We define the asymptotic dimension of a group G to be the asymptotic dimension of G with some left-invariant, proper metric. This is well-defined by Lemma 3.3 and Proposition 3.5.

For a metric space the definition of asymptotic dimension can be rephrased as follows.

Proposition 3.6 ([Roe03, Theorem 9.9])**.** *A metric space X has asymptotic dimension at most n if for every $r > 0$ there exists a bounded cover $\mathcal{U}$ of X of dimension at most n and Lebesgue-number at least r. That is,*

1. *every element $x \in X$ is contained in at most $n+1$ elements of $\mathcal{U}$;*
2. *for every $x \in X$ there exists $U \in \mathcal{U}$ with $B_r(x) \subseteq U$;*
3. $\sup_{U \in \mathcal{U}} \operatorname{diam} U < \infty$.

If a finite group F acts on X by isometries, then, given a cover $\mathcal{U}$ as in Proposition 3.6, we can consider the induced cover of $F \backslash X$. This is still bounded and has the same Lebesgue-number but the dimension is $|F|(n+1) - 1$. This yields the following result.

Corollary 3.7. *Let F be a finite group and let X be a metric space of asymptotic dimension at most n with an isometric F-action. Then $F\backslash X$ has asymptotic dimension at most $|F|(n+1)-1$.*

This corollary will be used for proving injectivity of the assembly map for groups with finite asymptotic dimension, see Theorem 4.3 below. It is also possible, see Theorem 9.4, to show that $F\backslash X$ has the same asymptotic dimension as X if X is proper.

4 Previous injectivity results

The general scheme of most of the injectivity results described here is the following. One constructs a map of controlled additive categories over EG such that one obtains the assembly map (for the trivial family) after taking K-theory and G-fixed points. Then, using geometric properties of G, one shows that the map is an equivalence after applying K-theory but before taking fixed points. Hence it is still an equivalence when applying homotopy fixed points. As a last step, one then compares fixed points with homotopy fixed points to obtain injectivity of the assembly map. For this certain finiteness properties of the classifying space have to be used. This idea was first used by Gunnar Carlsson and Erik Pedersen to prove the following result.

Theorem 4.1 ([CP95, Thm. A]). *Let G be a group with a finite model for the classifying space BG. Assume its universal cover EG admits a compactification X (meaning X is compact, and EG is an open dense subset) satisfying the following conditions:*

1. *the G-action extends to X;*
2. *X is metrizable;*
3. *X is contractible;*
4. *compact subsets of EG become small near $Y := X \setminus EG$, i.e. for every point $y \in Y$, every compact subset $K \subseteq EG$ and for every neighborhood U of y in X, there exists a neighborhood V of y in X such that if $gK \cap V \neq \emptyset$ for some $g \in G$, then $gK \subseteq U$.*

Then the assembly map in algebraic K-theory

$$\alpha\colon \operatorname*{colim}_{\mathrm{Or}_{\{1\}} G} \mathbf{K}_{\mathbf{A}} \to \mathbf{K}_{\mathbf{A}}(*)$$

admits a left inverse.

Note that the assumption that G admits a finite model for BG in particular implies that G is torsion-free. This result was generalized to groups with torsion by David Rosenthal. Here the scheme described above has to be slightly modified in the sense that now controlled categories over $\underline{E}G$ are considered and one has to show that the obtained map in K-theory is an equivalence when taking fixed points for any finite subgroup of G.

Theorem 4.2 ([Ros04, Theorem 6.1])**.** *Let G be a discrete group. Assume there exists a finite model for $\underline{E}G$ with a compactification X such that the following conditions hold:*

1. *the G-action extends to X;*
2. *X is metrizable;*
3. *X^H is contractible for every finite subgroup H of G;*
4. *$\underline{E}G^H$ is dense in X^H for every finite subgroup H of G;*
5. *compact subsets of $\underline{E}G$ become small near $X \setminus \underline{E}G$*

Then the assembly map in algebraic K-theory for the family of finite subgroups

$$\alpha_{\mathcal{F}in} \colon \operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in} G} \mathbf{K_A} \to \mathbf{K_A}(*)$$

admits a left inverse.

Arthur Bartels [Bar03b] as well as Gunnar Carlsson and Boris Goldfarb [CG05] proved a similar result for groups with finite asymptotic dimension. This was again for torsion-free groups and it was later generalized to groups with torsion by Bartels and Rosenthal as follows. This uses that $F\backslash G$ has again finite asymptotic dimension if G has this property.

Theorem 4.3 ([BR07b, Thm. A])**.** *Let G be a discrete group and let R be a ring. Assume that there is a finite G-CW-model for $\underline{E}G$ and that G has finite asymptotic dimension. Then the assembly map in algebraic K-theory for the family of finite subgroups*

$$\alpha_{\mathcal{F}in} \colon \operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in} G} \mathbf{K_A} \to \mathbf{K_A}(*)$$

admits a left inverse.

Remark 4.4. Note that [BR07b, Theorem A] was originally stated for groups with a finite dimensional model for $\underline{E}G$. But there was a gap in the proof of [BR07b, Proposition 7.5] such that only the above result is proved, see [BR17]. The results mentioned in Section 6 will however imply a version for groups that admit only a finite dimensional model.

The notion of finite asymptotic dimension was generalized by Guentner, Tessera and Yu. Before we introduce the concept of finite decomposition complexity and discuss injectivity results for groups with this property, we will end this section by mentioning another injectivity result from the literature.

Theorem 4.5 (Bökstedt–Hsiang–Madsen [BHM93])**.** *Let G be a group. Assume that the following condition holds:*

[A_1] For every $s \geq 1$ the integral group homology $H_s(BG;\mathbb{Z})$ is a finitely generated abelian group.

Then the assembly map

$$\alpha\colon \operatorname*{colim}_{\mathrm{Or}\,G} \mathbf{K}_{\mathbb{Z}}^{\geq 0} \to \mathbf{K}_{\mathbb{Z}}^{\geq 0}(*) \simeq \mathbf{K}^{\geq 0}(\mathbb{Z}[G])$$

is $\pi_*^{\mathbb{Q}}$-injective. Here $\mathbf{K}^{\geq 0}$ denotes the connective K-theory spectrum.

The next result stated below is a special case of [LRRV17, Main Technical Theorem 1.16] and generalizes Bökstedt–Hsiang–Madsen's Theorem.

Theorem 4.6 ([RV18, Thm. 69])**.** *Let G be a group and let $\mathcal{F}$ be a family of finite cyclic subgroups of G. Assume that the following two conditions hold.*

[$A_{\mathcal{F}}$] For every $C \in \mathcal{F}$ and every $s \geq 1$, the integral group homology $H_s(Z_G C;\mathbb{Z})$ of the centralizer of C in G is a finitely generated abelian group.

[$B_{\mathcal{F}}$] For every $C \in \mathcal{F}$ and every $t \geq 0$, the natural homomorphism

$$K_t(\mathbb{Z}[\xi_c]) \otimes_{\mathbb{Z}} \mathbb{Q} \to \prod_{p \ \textit{prime}} K_t(\mathbb{Z}_p \otimes \mathbb{Z}[\xi_c]; \mathbb{Z}_p) \otimes \mathbb{Q}$$

is injective, where c is the order of C, ξ_c is any primitive c-th root of unity, and $K_t(R;\mathbb{Z}_p) = \pi_t(\mathbf{K}(R)_p^{\wedge})$.

Then the assembly map for the family $\mathcal{F}$

$$\alpha_{\mathcal{F}}\colon \operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}}\,G} \mathbf{K}_{\mathbb{Z}}^{\geq 0} \to \mathbf{K}_{\mathbb{Z}}^{\geq 0}(*) \simeq \mathbf{K}^{\geq 0}(\mathbb{Z}[G])$$

is $\pi_*^{\mathbb{Q}}$-injective. Here $\mathbf{K}^{\geq 0}$ again denotes the connective K-theory spectrum.

This result differs from the previously stated results in two ways. It only holds rationally and it needs no geometric input on the group G. In the following sections, we will not discuss any results of this kind but focus on statements using the geometry of G.

5 Finite decomposition complexity

Definition 5.1. Let $\mathfrak{C}$ be a class of metric families. Let $n \in \mathbb{N}$ and $r > 0$. A metric family $\mathcal{X}$ is (r,n)-*decomposable* over $\mathfrak{C}$ if for every $X \in \mathcal{X}$ there is a decomposition $X = X_0 \cup X_1 \cup \cdots \cup X_n$ such that for each i, $0 \le i \le n$,

$$X_i = \bigsqcup_{r\text{-disjoint}} \{X_{ij} \mid j \in J_i\},$$

and the metric family $\{X_{ij} \mid X \in \mathcal{X},\ 0 \le i \le n,\ j \in J_i\}$ is in $\mathfrak{C}$.

The metric family $\mathcal{X}$ is n-*decomposable* over $\mathfrak{C}$ if $\mathcal{X}$ is (r,n)-decomposable over $\mathfrak{C}$ for every $r > 0$.

A metric family $\mathcal{X}$ is *strongly decomposable* over $\mathfrak{C}$ if it is 1-decomposable over $\mathfrak{C}$. It is *weakly decomposable* over $\mathfrak{C}$ if it n-decomposable over $\mathfrak{C}$ for some $n \in \mathbb{N}$.

Notice that the statement that $\mathcal{X}$ is n-*decomposable* over $\mathfrak{B}$ is precisely the statement that $\mathcal{X}$ has asymptotic dimension at most n.

Guentner, Tessera and Yu [GTY13] defined finite decomposition complexity as follows.

Definition 5.2. Let $\mathfrak{D}$ be the smallest class of metric families containing $\mathfrak{B}$ that is closed under strong decomposition, and let $w\mathfrak{D}$ be the smallest class of metric families containing $\mathfrak{B}$ that is closed under weak decomposition. A metric family in $\mathfrak{D}$ is said to have *finite decomposition complexity* (abbreviated to "FDC"), and a metric family in $w\mathfrak{D}$ is said to have *weak finite decomposition complexity* (abbreviated to "weak FDC"). As before, a metric space X has FDC or weak FDC if the metric family consisting only of X has this property.

We will focus on FDC and only briefly discuss weak FDC in Section 9. We will see in Section 6 that one can obtain injectivity results for assembly maps for groups with FDC. The reason that one has to restrict to groups with FDC rather than weak FDC is that the proof of these results relies on the use of Mayer-Vietoris type sequences and it seems unlikely that a generalization to more components is possible.

The advantage of FDC over finite asymptotic dimension is that it satisfies several inheritance properties as we will now discuss. This can be used to show that many groups have FDC. In particular, all linear groups and all elementary amenable groups have this property, see Corollary 5.9 and Theorem 5.14.

The first important property of FDC is that it is a coarse invariant.

Theorem 5.3 ([GTY13, Coarse Invariance 3.1.3])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families. If there is a coarse embedding from $\mathcal{X}$ to $\mathcal{Y}$ and $\mathcal{Y}$ has finite decomposition* *complexity, then so does $\mathcal{X}$.*

Using this and Proposition 3.5, we can say that a group G has FDC if it has FDC as a metric space with any proper left-invariant metric.

Finite groups obviously have FDC and if a group G has FDC, then it is not hard to show that $G \times \mathbb{Z}$ also has FDC. Hence we have the following example.

Example 5.4. All finite and all finitely generated abelian groups have FDC.

The following is probably the most important inheritance property of FDC. As we will see in Section 7 many other inheritance properties can be deduced from it.

Definition 5.5. A class $\mathfrak{C}$ of metric families satisfies *Fibering Permanence* if the following is satisfied. Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families and let $F\colon \mathcal{X} \to \mathcal{Y}$ be a uniformly expansive map. Assume $\mathcal{Y}$ is contained in $\mathfrak{C}$, and that for every bounded subfamily $\mathcal{Z}$ of $\mathcal{Y}$ the inverse image $F^{-1}(\mathcal{Z})$ is contained in $\mathfrak{C}$. Then $\mathcal{X}$ is contained in $\mathfrak{C}$.

Theorem 5.6 ([GTY13, Fibering Theorem 3.1.4])**.** *The class of metric families with FDC satisfies Fibering Permanence.*

While the class of groups with FDC satisfies more inheritance properties, we will not list all of them here but now give some of the inheritance properties of groups with FDC as these will allow us to give interesting examples of such groups.

Proposition 5.7 ([GTY13, Proposition 3.2.1])**.** *A countable direct union of groups with FDC has FDC. Equivalently, a countable discrete group has FDC if and only if every finitely generated subgroup does.*

Proposition 5.8 ([GTY13, Corollary 3.2.5])**.** *The class of countable discrete groups with FDC is closed under extensions.*

Combining Example 5.4 with Proposition 5.7 and Proposition 5.8, we obtain the following result.

Corollary 5.9 ([GTY13, Theorem 5.1.2])**.** *Elementary amenable groups have FDC.*

Using Bass-Serre theory one can also prove the following inheritance result.

Proposition 5.10 ([GTY13, Proposition 3.2.6])**.** *If a countable discrete group acts (without inversion) on a tree, and the vertex stabilizers of the action have FDC, then the group itself has FDC.*

We will now sketch the proof that spaces, and thus groups, with finite asymptotic dimension have FDC.

Theorem 5.11 ([Kas, Theorem 1])**.** *Let X be a metric space with asymptotic dimension at most n. Then there exists a coarse embedding of X into a product of $n+1$ trees.*

The following corollary is obtained by decomposing one tree at a time.

Corollary 5.12. *Let X be a metric space with finite asymptotic dimension. Then X has FDC.*

Theorem 5.11 was proved by Dranishnikov for geodesic metric spaces of bounded geometry [Dra03, Theorem 3]. Dranishnikov and Zarichnyi [DZ04, Theorem 3.5] showed that proper metric spaces can be coarsely embedded into a product of $n+1$ binary trees. Using an ultralimit construction, Guentner, Tessera and Yu [GTY13, Proof of Theorem 4.1] used this to show that every metric space with asymptotic dimension at most n can be embedded into a product of $(n+1)$-many 0-hyperbolic spaces. This suffices to show that metric families with finite asymptotic dimension have FDC. But there is a simple and direct proof of Theorem 5.11 without using ultralimits. The main ingredient is the following lemma. Its proof is an elementary rearranging of the involved covers.

Let X be a metric space of asymptotic dimension at most n and let f' be a non-decreasing control function. Define $f(x) := f'(3x) + 3x$ and define $g\colon \mathbb{N} \to \mathbb{R}$ inductively by $g(0) = 2$ and

$$g(k) := 100f(g(k-1)).$$

Lemma 5.13 ([Kas, Lemma 5])**.** *There exist $\frac{9}{10}g(k)$-disjoint covers $\mathcal{U}_k^0, \ldots, \mathcal{U}_k^n$ of diameter at most $2f(g(k))$ such that*

1. *for every j, every $l < k$ and all $U \in \mathcal{U}_k^j, V \in \mathcal{U}_l^j$ we have $d(U,V) \leq l$ implies $V^l \subseteq U$;*
2. *for $0 \leq i \leq n$ with $i = k$ modulo $n+1$, we have $B_k(x_0) \subseteq U$ for some $U \in \mathcal{U}_k^i$.*

Given covers $\mathcal{U}_k^0, \ldots, \mathcal{U}_k^n$ as in Lemma 5.13, we define trees T^j as follows. The vertices of T^j are the elements $\{U \mid k \in \mathbb{N}, U \in \mathcal{U}_k^j\}$ and $V \in \mathcal{U}_k^j, U \in \mathcal{U}_{k'}^j$ with $k < k'$ are connected by an edge if and only if $V \subseteq U$ and there is no $k < l < k'$ such that there exists $W \in \mathcal{U}_l^j$ with $V \subseteq W$. Note that this in particular implies that V and U are connected by a sequence of edges if $V \subseteq U$.

For $x \in X$ define $\psi^j(x) := \min\{k \in \mathbb{N} \mid \exists U \in \mathcal{U}_k^j, x \in U\}$ and let $\varphi^j(x) \in \mathcal{U}_{\psi^j(x)}^j$ be the element containing x.

It is then straightforward to prove that the map $\varphi = \prod_{j=0}^n \varphi^j \colon X \to \prod_{j=0}^n T^j$ is a coarse embedding as required.

We end this section by mentioning the following result.

Theorem 5.14 ([GTY13, Theorem 5.2.2])**.** *Let R be a commutative ring with unit. Every countable subgroup of $GL_n(R)$ has FDC.*

The main ingredients in the proof are the Fibering Theorem (Theorem 5.6) and the following proposition.

Proposition 5.15 ([GTY12, Proposition 3.8])**.** *Let γ be an archimedean or a discrete norm on a field K. The group $GL_n(K)$, equipped with the left-invariant pseudometric induced by γ, has finite asymptotic dimension.*

6 Injectivity for groups with FDC

Using the methods from [GTY12], Ramras, Tessera and Yu proved the following result.

Theorem 6.1 ([RTY14, Theorem 1.1])**.** *Let G be a discrete group with finite decomposition complexity and assume that BG admits a finite CW-model. Then the assembly map in algebraic K-theory*

$$\alpha\colon \operatorname*{colim}_{\mathrm{Or}_{\{1\}} G} \mathbf{K}_{\mathbf{A}} \to \mathbf{K}_{\mathbf{A}}(*)$$

admits a left inverse.

As before the assumption that BG admits a finite CW-model in particular implies that G is torsion-free. In my PhD thesis I generalized this result to groups with a finite dimensional model for $\underline{E}G$. This not only allows to consider groups with torsion but also weakens the finiteness assumption from requiring a finite model to a finite-dimensional one. However, in this case it is no longer sufficient if G itself has FDC but it has to have this property equivariantly for all finite subgroups. More precisely, I proved the following statement.

Theorem 6.2 ([Kas14, Thm. 3.2.2])**.** *Let G be a discrete group such that the family $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC. Assume that there is a finite dimensional G-CW-model for the classifying space for proper actions $\underline{E}G$. Then the assembly map in algebraic K-theory for the family of finite subgroups*

$$\alpha_{\mathcal{F}in}\colon \operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in} G} \mathbf{K}_{\mathbf{A}} \to \mathbf{K}_{\mathbf{A}}(*)$$

admits a left inverse.

This result generalizes Theorem 4.3 by Bartels and Rosenthal which can be seen as follows. If G admits a finite G-CW-model for $\underline{E}G$, there is an upper bound on the order of the finite subgroups of G. In this case, Corollary 3.7 can be used to show that $\{H\backslash G\}_{H\in\mathcal{F}in}$ has finite asymptotic dimension if G has finite asymptotic dimension.

In general, determining whether the family $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC is much harder than only showing that G has FDC. The main reason is that it is unclear whether similar inheritance properties as for those of groups with FDC hold. The problem here is that for example in an extension of groups new finite subgroups can appear that have to be considered. Before discussing this problem further in Section 9, we will in the next section introduce a slight modification of FDC that allows to circumvent this problem.

In Section 8 we will see that Theorem 6.2 can be used to deduce injectivity results for a large class of linear groups and Lie groups.

7 Regular finite decomposition complexity

Inspired by the work of Guentner, Tessera, and Yu, the notion of *regular finite decomposition complexity* was introduced in joint work with Andrew Nicas and David Rosenthal in [KNR].

Definition 7.1 ([KNR, Def. 2.6])**.** A metric family $\mathcal{X}$ *regularly decomposes* over a class of metric families $\mathfrak{C}$ if there exists a family $\mathcal{Y}$ with finite asymptotic dimension and a coarse map $F\colon \mathcal{X} \to \mathcal{Y}$ such that for every bounded subfamily $\mathcal{B}$ of $\mathcal{Y}$, the inverse image $F^{-1}(\mathcal{B})$ lies in $\mathfrak{C}$.

Definition 7.2 ([KNR, Def. 2.7])**.** Let $\mathfrak{R}$ be the smallest class of metric families containing $\mathfrak{B}$ that is closed under regular decomposition. A metric family in $\mathfrak{R}$ is said to have *regular finite decomposition complexity* (abbreviated to "regular FDC").

It is straightforward to see that every metric family with finite asymptotic dimension has regular FDC. More precisely, we have the following statement.

Theorem 7.3 ([KNR, Thm. 5.3])**.** *The class of metric families with regular FDC is the smallest class of metric families that contains all families with finite asymptotic dimension and satisfies Fibering Permanence.*

Since all families with finite asymptotic dimension have FDC by Corollary 5.12 and the class of metric families with FDC satisfies Fibering Permanence by Theorem 5.6, we have the following corollary.

Corollary 7.4. *Every metric family with regular FDC has FDC.*

While we do not know whether the converse is also true, the class $\mathfrak{R}$ of metric families with regular FDC satisfies all the known inheritance properties of $\mathfrak{D}$. More generally, every class that satisfies Fibering Permanence and contains all metric families with finite asymptotic dimension has these inheritance properties, see [KNR, Theorem 1.1]. In particular, we have the following result.

Theorem 7.5 ([KNR, Cor. 1.2])**.** *The class of (countable) groups with regular FDC is closed under extensions, direct unions, free products (with amalgam) and relative* *hyperbolicity. Furthermore, all elementary amenable groups, all linear groups and all subgroups of virtually connected Lie groups have regular FDC.*

The property that makes the notion of regular FDC particularly useful for our purposes is that is closed under taking quotients by finite groups. That is, it satisfies the following.

Theorem 7.6 ([KNR, Cor. 5.29])**.** *Let G be a countable group that has regular FDC and a global upper bound on the orders of its finite subgroups. Then the metric family $\{F\backslash G \mid F \leq G \text{ finite}\}$ has regular FDC.*

Combining this result with Theorem 6.2, we obtain the following.

Corollary 7.7. *Let G be a group with regular FDC. Assume there exists a finite dimensional model for $\underline{E}G$ and a global upper bound on the order of the finite subgroups of G. Then the assembly map in algebraic K-theory for the family of finite subgroups*

$$\alpha_{\mathcal{F}in}\colon \operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}G} \mathbf{K}_\mathbf{A} \to \mathbf{K}_\mathbf{A}(*)$$

admits a left inverse.

As mentioned in Theorem 7.5, all linear groups have regular FDC. In particular, they satisfy the assumptions of the corollary if there exists a finite dimensional model for $\underline{E}G$ and a global upper bound on the order of the finite subgroups. We will show in the next section that the latter assumption is not needed for linear groups and also discuss the first assumption.

8 Linear groups and Lie groups

From Theorem 7.5 and Corollary 7.7 we can deduce the following result. The assumption on the global upper bound on the orders of the finite subgroups holds by Selberg's Lemma [Sel60].

Theorem 8.1 ([Kas15a, Corollary 3])**.** *Let $\mathbb{F}$ be a field of characteristic zero. Let G be a finitely generated subgroup of a $GL_n(\mathbb{F})$, and assume there exists a finite-dimensional model for $\underline{E}G$. Then the assembly map in algebraic K-theory for the family of finite subgroups*

$$\alpha_{\mathcal{F}in}\colon \operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}G} \mathbf{K}_\mathbf{A} \to \mathbf{K}_\mathbf{A}(*)$$

admits a left inverse.

Given a finitely generated subgroup of a virtually connected Lie group we can consider its adjoint representation to obtain a homomorphism to $GL_n(\mathbb{C})$ with abelian kernel. Using this together with the Farrell–Jones conjecture for virtually solvable groups [Weg15] and an inheritance result for injectivity [Kas15b, Proposition 4.1] yields the following corollary of Theorem 8.1.

Theorem 8.2 ([Kas15b, Theorem 1.1])**.** *Let G be a finitely generated subgroup of a virtually connected Lie group, and assume there exists a finite-dimensional model for $\underline{E}G$. Then the assembly map in algebraic K-theory for the family of finite subgroups*

$$\alpha_{\mathcal{F}in}\colon \operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in} G} \mathbf{K_A} \to \mathbf{K_A}(*)$$

admits a left inverse.

The use of the Farrell–Jones conjecture to prove Theorem 8.2 can be avoided by using an injectivity result for a relative assembly map instead, see Section 10.

The assumption that the field $\mathbb{F}$ in Theorem 8.1 has characteristic zero can be removed to obtain the following result.

Theorem 8.3 ([Kas16, Theorem 1.1])**.** *Let R be a commutative ring with unit. Let G be a finitely generated subgroup of a $GL_n(R)$, and assume there exists a finite-dimensional model for $\underline{E}G$. Then the assembly map in algebraic K-theory for the family of finite subgroups*

$$\alpha_{\mathcal{F}in}\colon \operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in} G} \mathbf{K_A} \to \mathbf{K_A}(*)$$

admits a left inverse.

The main obstacle in the proof is that for fields of positive characteristic finitely generated linear groups are not necessarily virtually torsion-free. The key idea to overcome this problem is to use a version of Selberg's lemma that states that there is a finite-index subgroup such that all its finite subgroups are unipotent. Now one can use that the subgroup of unipotent upper triangular matrices has asymptotic dimension zero and that every quotient of a space of asymptotic dimension zero still has asymptotic dimension zero. To pass from fields to general commutative rings, one first notices that if the ring R has trivial nilradical, then the linear group over R embeds into a product over linear groups over fields. By taking the quotient by the nilradical one sees that the linear group over a general commutative ring sits in an extension of a linear group over a ring with trivial nilradical and a nilpotent group.

We end this section by stating the following theorem regarding the existence of finite dimensional models for $\underline{E}G$.

Theorem 8.4 ([Kas15b, Proposition 1.3] and [Kas16, Proposition 1.2])**.** *Let R be a commutative ring with unit and let G be a finitely generated subgroup of $GL_n(R)$ or of a virtually connected Lie group. Then G admits a finite-dimensional model for*

$\underline{E}G$ if and only if there exists a global upper bound on the Hirsch rank of the solvable subgroups of G.

The previously mentioned injectivity results, in particular those of this section, have the following analog version in L-theory. Here the extra condition on the negative K-groups comes in since it is needed so that L-theory commutes with infinite products, see [Win13]. This property is used when comparing fixed points with homotopy fixed points.

Theorem 8.5 ([Kas15b, Theorem 6.1],[Kas16, Theorem 1.1],[KNR, Theorem 1.3])**.** *Let G be a group satisfying one of the following conditions.*

1. *G is a finitely generated subgroup of a virtually connected Lie group;*
2. *G is a finitely generated linear group over a commutative ring R or*
3. *G has regular FDC and a global upper bound on the order of its*

   finite subgroups.

Assume that there exists a finite dimensional model for $\underline{E}G$. Let $\mathbf{A}$ be an additive G-category with involution. Assume further that for every finite subgroup F of G there is an $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$ we have $K_{-i}(\mathbf{A} *_F F/F) = 0$. Then the L–theoretic assembly map

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in} G} \mathbf{L}_{\mathbf{A}}^{\langle\infty\rangle} \to \mathbf{L}_{\mathbf{A}}^{\langle\infty\rangle}(*)$$

admits a left inverse.

9 Some open problems

In this section, we want to discuss some open questions regarding finite asymptotic dimension and finite decomposition complexity.

As we have seen regular finite decomposition complexity implies finite decomposition complexity and finite decomposition complexity implies weak finite decomposition complexity by definition. For both these implications it is not known whether the converse holds, i.e. we have the following questions.

Question 9.1.

1. Does every metric family with FDC also have regular FDC?
2. Does every metric family with weak FDC also have FDC?

Arguing as in the proof of Corollary 3.7, if a metric space X has weak FDC and a finite group F acts on X by isometries, then $F\backslash X$ has weak FDC. The same is true in the case of regular FDC by Theorem 7.6. Hence closely connected to the above question it the following one.

Question 9.2. Given a metric space X with FDC and an isometric action by a finite group F, has $F\backslash X$ FDC?

A positive answer to Question 9.1.(1) or Question 9.1.(2) implies a positive answer to Question 9.2. And a positive answer to Question 9.2 would imply that in Corollary 7.7 regular FDC can be replaced by FDC. Since there are no groups known that have FDC but for which regular FDC is unknown, this generalization would not lead to any new application directly. A more interesting question is the following.

Question 9.3. Given a group G with finite asymptotic dimension, has $\{H\backslash G\}_{H\in\mathcal{F}in}$ finite asymptotic dimension?

By Corollary 3.7, the answer to Question 9.3 is positive for groups which have a global upper bound on the order of their finite subgroups. As we have seen in Section 8 (see also [Kas16, Theorem 3.3]), the answer is also positive for linear groups over fields of positive characteristic although those have no such upper bound in general.

In view of Corollary 3.7, one might attempt to construct a counterexample to Question 9.3 in the following way. Try to find groups G with finite subgroups F such that the asymptotic dimension of $F\backslash G$ is larger than the asymptotic dimension of G roughly by a factor of $|F|$ and then combine these into an actual counterexample. The following theorem shows that already the first step of this idea fails.

Theorem 9.4 ([Kas17, Theorem 1.1])**.** *Let X be a proper metric space and let F be a finite group acting isometrically on X. Then $F\backslash X$ has the same asymptotic dimension as X.*

10 Coarse homology theories

In this section with describe a generalization of the proof of the previous injectivity results. As mentioned before the central idea is to consider controlled categories over classifying spaces. More generally, one can consider controlled categories over any bornological coarse space and thus obtain a coarse homology theory. In joint work with Ulrich Bunke, Alexander Engel and Christoph Winges, we considered this setting more axiomatically and obtained properties of general coarse homology theories that allowed us to prove similar injectivity results. We first give the necessary background. Most of this material has been developed in [BEKWa] (see also [BE] for the non-equivariant case).

Definition 10.1. Let G be a group and let X be a G-set. For a subset U of the power set $\mathcal{P}(X \times X)$ of $X \times X$ we define the *inverse* by

$$U^{-1} := \{(y, x) \mid (x, y) \in U\}$$

and for $U, V \in \mathcal{P}(X \times X)$ we define their *composition* by

$$U \circ V := \{(x, z) \mid \exists y \in X : (x, y) \in U \wedge (y, z) \in V\}.$$

Definition 10.2. A *G-coarse structure* $\mathcal{C}$ on X is a subset of $\mathcal{P}(X \times X)$ with the following properties:

1. $\mathcal{C}$ is closed under composition, inversion, and forming finite unions or subsets.
2. $\mathcal{C}$ contains the diagonal $\Delta(X)$ of X.
3. For every $U \in \mathcal{C}$, the set GU is also in $\mathcal{C}$.

The pair $(X, \mathcal{C})$ is called a *G-coarse space*, and the members of $\mathcal{C}$ are called (coarse) *entourages* of X.

Definition 10.3. Let $(X, \mathcal{C})$ and $(X', \mathcal{C}')$ be G-coarse spaces and let $f \colon X \to X'$ be an equivariant map between the underlying sets. The map f is *controlled* if for every $U \in \mathcal{C}$ we have $(f \times f)(U) \in \mathcal{C}'$.

We obtain a category of G-coarse spaces and controlled equivariant maps.

Definition 10.4. A *G-bornology* $\mathcal{B}$ on X is a subset of $\mathcal{P}(X)$ with the following properties:

1. $\mathcal{B}$ is closed under forming finite unions and subsets.
2. $\mathcal{B}$ contains all finite subsets of X.
3. $\mathcal{B}$ is G-invariant.

The pair $(X, \mathcal{B})$ is called a *G-bornological space*, and the members of $\mathcal{B}$ are called *bounded subsets* of X.

Definition 10.5. Let $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ be G-bornological spaces and let $f \colon X \to X'$ be an equivariant map between the underlying sets. The map f is *proper* if for every $B' \in \mathcal{B}'$ we have $f^{-1}(B') \in \mathcal{B}$.

Definition 10.6. Let X be a G-set with a G-coarse structure $\mathcal{C}$ and a G-bornology $\mathcal{B}$. The coarse structure $\mathcal{C}$ and the bornology $\mathcal{B}$ are said to be *compatible* if for every $B \in \mathcal{B}$ and $U \in \mathcal{C}$ the *U-thickening* $U[B] := \{x \in X \mid \exists y \in B : (x, y) \in U\}$ lies in $\mathcal{B}$.

Definition 10.7. A *G-bornological coarse space* is a triple $(X, \mathcal{C}, \mathcal{B})$ consisting of a G-set X, a G-coarse structure $\mathcal{C}$, and a G-bornology $\mathcal{B}$ such that $\mathcal{C}$ and $\mathcal{B}$ are compatible.

Definition 10.8. A *morphism* $f\colon (X,\mathcal{C},\mathcal{B}) \to (X',\mathcal{C}',\mathcal{B}')$ between G-bornological coarse spaces is an equivariant map $f\colon X \to X'$ of the underying G-sets which is controlled and proper.

We obtain a category $G\mathbf{BC}$ of G-bornological coarse spaces and morphisms. If the structures are clear from the context, we will use the notation X instead of $(X,\mathcal{C},\mathcal{B})$ in order to denote G-bornological coarse spaces.

We now introduce the notion of an equivariant coarse homology theory, see [BEKWa, Section 3] for details.

Definition 10.9. Let X be a G-bornological coarse space.

An *equivariant big family* on X is a filtered family of G-invariant subsets $(Y_i)_{i\in I}$ of X such that for every entourage U of X and $i \in I$ there exists $j \in I$ such that $U[Y_i] \subseteq Y_j$.

An *equivariant complementary pair* $(Z,\mathcal{Y})$ on X is a pair of a G-invariant subset Z of X and an equivariant big family $\mathcal{Y} = (Y_i)_{i\in I}$ on X such that there exists $i \in I$ with $Z \cup Y_i = X$.

Definition 10.10. Let X be a G-bornological coarse space. The space X is *flasque* if it admits a morphism $f\colon X \to X$ such that:

1. f is close to id_X, i.e., $(f,\mathrm{id}_X)(\Delta(X))$ is an entourage of X.
2. For every entourage U of X the subset $\bigcup_{n\in\mathbb{N}}(f^n \times f^n)(U)$ is an entourage of X.
3. For every bounded subset B of X there exists an integer n such that $B \cap f^n(X) = \emptyset$.

Let $\mathbf{C}$ be a cocomplete stable ∞-category and let

$$E\colon G\mathbf{BC} \to \mathbf{C}$$

be a functor. If $\mathcal{Y} = (Y_i)_{i\in I}$ is a filtered family of G-invariant subsets of X, then we set

$$E(\mathcal{Y}) := \operatorname*{colim}_{i\in I} E(Y_i).$$

In this formula we consider the subsets Y_i as G-bornological coarse spaces with the structures induced from X.

Definition 10.11. Let $\mathbf{C}$ be a cocomplete stable ∞-category. A functor

$$E\colon G\mathbf{BC} \to \mathbf{C}$$

is called a *G-equivariant $\mathbf{C}$-valued coarse homology theory* if it satisfies the following conditions:

1. (Coarse invariance) For all $X \in G\mathbf{BC}$ the functor E sends the projection $I \otimes X \to X$ to an equivalence, where I denotes the bornological coarse space consisting of the set $\{0,1\}$ with the maximal coarse structure generated by the whole set $\{0,1\} \times \{0,1\}$ and the maximal bornology generated by the whole set $\{0,1\}$.

2. (Excision) $E(\emptyset) \simeq 0$ and for every equivariant complementary pair $(Z, \mathcal{Y})$ on a G-bornological coarse space X the square

$$\begin{tikzcd} E(Z \cap \mathcal{Y}) \arrow[r] \arrow[d] & E(Z) \arrow[d] \\ E(\mathcal{Y}) \arrow[r] & E(X) \end{tikzcd}$$

is a push-out.

3. (Flasqueness) If a G-bornological coarse space X is flasque, then $E(X) \simeq 0$.

4. (u-Continuity) For every G-bornological coarse space X the natural map

$$\operatorname*{colim}_{U \in \mathcal{C}^G} E(X_U) \to E(X)$$

is an equivalence. Here X_U denotes the G-bornological coarse space X with the coarse structure replaced by the one generated by U.

If the group G is clear from the context, then we will often just speak of an equivariant coarse homology theory.

We have a universal equivariant coarse homology theory

$$\mathrm{Yo}^s \colon G\mathbf{BC} \to G\mathbf{Sp}\mathcal{X}$$

(see [BEKWa, Definition 4.9]), where $G\mathbf{Sp}\mathcal{X}$ is a stable presentable ∞-category called the category of coarse motivic spectra. More precisely, for every cocomplete stable ∞-category $\mathbf{C}$ we have the following.

Proposition 10.12 ([BEKWa, Corollary 4.10]). *Restriction along* Yo^s *induces an equivalence between the ∞-categories of colimit-preserving functors* $G\mathbf{Sp}\mathcal{X} \to \mathbf{C}$ *and* $\mathbf{C}$*-valued equivariant coarse homology theories.*

The category $G\mathbf{BC}$ has a symmetric monoidal structure $\otimes$, see [BEKWa, Example 2.17].

Let $E \colon G\mathbf{BC} \to \mathbf{C}$ be a functor and let X be a G-bornological coarse space.

Definition 10.13. The *twist* E_X of E by X is the functor

$$E(X \otimes -)\colon G\mathbf{BC} \to \mathbf{C}.$$

Lemma 10.14. *If E is an equivariant coarse homology theory, then the twist E_X is again an equivariant coarse homology theory.*

Proof. This follows from [BEKWa, Lemma 4.17]. □

We need various additional properties or structures for an equivariant coarse homology theory.

1. The property of *continuity* of an equivariant coarse homology theory was defined in [BEKWa, Definition 5.15].
2. The property of *strong additivity* of an equivariant coarse homology theory was defined in [BEKWa, Definition 3.12].
3. The additional structure of transfers for an equivariant coarse homology theory is encoded in the notion of a *coarse homology theory with transfers* which was defined in [BEKWc].

We can embed the orbit category $\mathrm{Or}\, G$ into $G\mathbf{BC}$ by a functor

$$i\colon \mathrm{Or}\, G \to G\mathbf{BC}$$

which sends a transitive G-set S to the G-bornological coarse space $S_{min,max}$ with underlying G-set S, the minimal coarse structure generated by the diagonal $\Delta(S)$ and the maximal bornology generated by the whole set S.

Let $G_{can,min}$ denote the G-bornological coarse space consisting of G with the canonical coarse and the minimal bornological structures, i.e. with the coarse structure $\mathcal{C}$ generated by the sets $B \times B$ for all finite subsets $B \subseteq G$ and the bornology $\mathcal{B}$ consisting of all finite subsets $B \subseteq G$.

Definition 10.15. We call $M\colon \mathrm{Or}\, G \to \mathbf{C}$ a *CP-functor* if it satisfies the following assumptions:

1. $\mathbf{C}$ is stable, complete, cocomplete, and compactly generated;
2. There exists an equivariant coarse homology theory E satisfying:
   (a) M is equivalent to $E_{G_{can,min}} \circ i$;
   (b) E is strongly additive;
   (c) E is continuous;

(d) E extends to a coarse homology theory with transfers.

We will consider the following families of subgroups.

Definition 10.16.

1. $\mathcal{FDC}$ denotes the family of subgroups V of G such that $\{F\backslash V\}_{F\in\mathcal{F}in(V)}$ has FDC, where $\mathcal{F}in(V)$ denotes the family of finite subgroups of V.
2. **cp** denotes the family of subgroups of G generated by those subgroups V such that $\underline{E}V$ admits a finite V-CW-model.
3. $\mathcal{FDC}^{\mathbf{cp}}$ denotes the intersection of $\mathcal{FDC}$ and **cp**.

We can now state our main theorem about injectivity results for coarse homology theories.

Theorem 10.17 ([BEKWb, Theorem 1.11]). *Let G be a group and let $\mathcal{F}$ be a family of subgroups. Assume that $M\colon \operatorname{Or} G \to \mathbf{C}$ is a CP-functor. Furthermore, assume that one of the following conditions holds:*

1. *$\mathcal{F}$ is a subfamily of $\mathcal{FDC}^{\mathbf{cp}}$ such that $\mathcal{F}in \subseteq \mathcal{F}$;*
2. *$\mathcal{F}$ is a subfamily of $\mathcal{FDC}$ such that $\mathcal{F}in \subseteq \mathcal{F}$ and G admits a finite-dimensional model for $E_{\mathcal{F}in}G$.*

Then the relative assembly map

$$\alpha^{\mathcal{F}}_{\mathcal{F}in,M}\colon \operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}in} G} M \to \operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}} G} M$$

admits a left inverse.

The following are our main examples of CP-functors.

Example 10.18.

1. The equivariant K-theory functor

   $$\mathbf{K}_{\mathbf{A}}\colon \operatorname{Or} G \to \mathbf{Sp}$$

   is an example of a CP-functor. This can be seen as follows. By [BEKWa, Cororllary 8.25], we have an equivalence

   $$K_{\mathbf{A}} \simeq K\mathbf{A}\mathcal{X}^G_{G_{can,min}} \circ i,$$

   where $K\mathbf{A}\mathcal{X}^G : G\mathbf{BC} \to \mathbf{Sp}$ denotes the coarse algebraic K-homology functor. By [BEKWc, Theorem 1.4], the functor $K\mathbf{A}\mathcal{X}^G$ admits an extension to an equivariant coarse homology theory with transfers. Furthermore, $K\mathbf{A}\mathcal{X}^G$ is continuous by [BEKWa, Proposition 8.17] and strongly additive by [BEKWa, Proposition 8.19].

2. For a group G, let P be the total space of a principal G-bundle and let $\mathbf{A}$ denote the functor of nonconnective A-theory (taking values in the ∞-category of spectra). Then P gives rise to an $\operatorname{Or} G$-spectrum $\mathbf{A}_P$ sending a transitive G-set S to the spectrum $\mathbf{A}(P \times_G S)$. By [BKW, Theorem 5.17], $\mathbf{A}_P$ is a CP-functor.

From Theorem 10.17 we can deduce the next two corollaries. For algebraic K-theory the first corollary was proved by Bartels [Bar03a] and the second corollary is Theorem 6.2.

Corollary 10.19 ([BEKWb, Corollary 1.13]). *Let G be a group. If $M\colon \operatorname{Or} G \to \mathbf{C}$ is a CP-functor, then the relative assembly map*

$$\alpha^{\mathcal{VC}yc}_{\mathcal{F}in,M}\colon \operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}in} G} M \to \operatorname*{colim}_{\operatorname{Or}_{\mathcal{VC}yc} G} M$$

admits a left inverse.

Corollary 10.20 ([BEKWb, Corollary 1.14]). *Let G be a group and assume that:*

1. *$M\colon \operatorname{Or} G \to \mathbf{C}$ is a CP-functor;*

2. *G admits a finite-dimensional model for $\underline{E}G$;*

3. *$\{F\backslash G\}_{F \in \mathcal{F}in}$ has FDC.*

Then the assembly map for the family of finite subgroups

$$\alpha_{\mathcal{F}in,M}\colon \operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}in} G} M \to M(*)$$

admits a left inverse.

As an application of Theorem 10.17 we also obtain the following new injectivity result for algebraic K-theory.

Theorem 10.21 ([BEKWb, Theorem 1.15]). *Suppose G admits a finite-dimensional model for $\underline{E}G$ and is relatively hyperbolic to groups $P_1, \dots, P_n$. Assume that each P_i is contained in $\mathcal{FDC}$ or satisfies the K-theoretic Farrell–Jones conjecture. Then the assembly map in algebraic K-theory for the family of finite subgroups*

$$\alpha_{\mathcal{F}in}\colon \operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}in} G} \mathbf{K}_{\mathbf{A}} \to \mathbf{K}_{\mathbf{A}}(*)$$

admits a left inverse.

As already mentioned, Theorem 10.17 can also be used to prove Theorem 8.2 without using the Farrell–Jones conjecture for virtually solvable groups. This was carried out in [BEKWb, Section 2].

11 K-theory as a coarse homology theory

As mentioned in Example 10.18, the algebraic K-theory of an additive category forms a CP-functor. For this, one of the more difficult properties to prove is strong additivity of the associated coarse homology theories since this relies on the fact that algebraic K-theory commutes with infinite products of additive categories. This was first proved by Carlsson [Car95] and uses for the most part simplicial techniques involving what he calls quasi-Kan complexes. Using Grayson's model [Gra12] of higher algebraic K-theory via binary acyclic complexes, in joint work with Christoph Winges [KW] we gave a new proof of this result. The advantage of this approach is that is not only elementary, but also exhibits the result as a consequence of the universal property of algebraic K-theory. For this reason the proof can be adapted to the setting of stable ∞-categories instead of additive categories. This was carried out in [KW19]. More precisely, we first carried out Grayson's approach in the setting of stable ∞-categories and showed that again higher K-groups can be described using (cubes of) binary acyclic complex. We then proved the following theorem by showing that one can restrict to complexes of a fixed length instead of considering arbitrary bounded complexes.

Theorem 11.1 ([KW19, Theorem 1.3]). *For every family $\{\mathcal{C}_i\}_{i\in I}$ of small stable ∞-categories, the natural map*

$$\mathbf{K}\left(\prod_{i\in I}\mathcal{C}_i\right) \to \prod_{i\in I}\mathbf{K}(\mathcal{C}_i)$$

is an equivalence.

This result was applied in [BKW] for proving that A-theory yields a CP-functor.

References

[Bar03a] Arthur C. Bartels, *On the domain of the assembly map in algebraic K-theory*, Algebr. Geom. Topol. **3** (2003), 1037–1050.

[Bar03b] ———, *Squeezing and higher algebraic K-theory*, K-Theory **28** (2003), no. 1, 19–37.

[BB19] Arthur Bartels and Mladen Bestvina, *The Farrell-Jones conjecture for mapping class groups.*, Invent. Math. **215** (2019), no. 2, 651–712.

[BE] Ulrich Bunke and Alexander Engel, *Homotopy theory with bornological coarse spaces*, arXiv:1607.03657.

[BEKWa] Ulrich Bunke, Alexander Engel, Daniel Kasprowski, and Christoph Winges, *Equivariant coarse homotopy theory and coarse algebraic K-homology*, arXiv:1710.04935.

[BEKWb] ______, *Injectivity results for coarse homology theories*, arXiv:1809.11079.

[BEKWc] ______, *Transfers in coarse homology*, arXiv:1809.08300.

[BHM93] M. Bökstedt, W. C. Hsiang, and I. Madsen, *The cyclotomic trace and algebraic K-theory of spaces.*, Invent. Math. **111** (1993), no. 3, 465–539.

[BKW] Ulrich Bunke, Daniel Kasprowski, and Christoph Winges, *Split injectivity of A-theoretic assembly maps*, arXiv:1811.11864.

[BL12] Arthur Bartels and Wolfgang Lück, *The Borel conjecture for hyperbolic and* CAT(0)*-groups*, Ann. of Math. (2) **175** (2012), no. 2, 631–689.

[BLR08] Arthur Bartels, Wolfgang Lück, and Holger Reich, *The K-theoretic Farrell-Jones conjecture for hyperbolic groups*, Invent. Math. **172** (2008), no. 1, 29–70.

[BR07a] Arthur Bartels and Holger Reich, *Coefficients for the Farrell-Jones conjecture*, Adv. Math. **209** (2007), no. 1, 337–362.

[BR07b] Arthur Bartels and David Rosenthal, *On the K-theory of groups with finite asymptotic dimension*, J. Reine Angew. Math. **612** (2007), 35–57.

[BR17] ______, *Erratum to: On the K-theory of groups with finite asymptotic dimension (J. reine angew. Math. 612 (2007), 35–57)*, J. Reine Angew. Math. **726** (2017), 291–292.

[Car95] Gunnar Carlsson, *On the algebraic K-theory of infinite product categories*, K-Theory **9** (1995), no. 4, 305–322.

[CG05] Gunnar Carlsson and Boris Goldfarb, *The integral K-theoretic Novikov conjecture for groups with finite asymptotic dimension.*, Invent. Math. **157** (2005), no. 2, 405–418.

[CP95] Gunnar Carlsson and Erik Kjær Pedersen, *Controlled algebra and the Novikov conjectures for K- and L-theory*, Topology **34** (1995), no. 3, 731–758.

[Dra03] A. N. Dranishnikov, *On hypersphericity of manifolds with finite asymptotic dimension.*, Trans. Am. Math. Soc. **355** (2003), no. 1, 155–167.

[DZ04] A. Dranishnikov and M. Zarichnyi, *Universal spaces for asymptotic dimension.*, Topology Appl. **140** (2004), no. 2-3, 203–225.

[Gra12] Daniel R. Grayson, *Algebraic K-theory via binary complexes*, J. Amer. Math. Soc. **25** (2012), no. 4, 1149–1167.

[GTY12] Erik Guentner, Romain Tessera, and Guoliang Yu, *A notion of geometric complexity and its application to topological rigidity*, Invent. Math. **189** (2012), no. 2, 315–357.

[GTY13] ———, *Discrete groups with finite decomposition complexity*, Groups Geom. Dyn. **7** (2013), no. 2, 377–402.

[Kas] Daniel Kasprowski, *Coarse embeddings into products of trees*, arXiv:1810.13361.

[Kas14] ———, *On the K-theory of groups with finite decomposition complexity*, Ph.D. thesis, Westfälische Wilhelms Universität Münster, 2014.

[Kas15a] ———, *On the K-theory of groups with finite decomposition complexity*, Proceedings of the London Mathematical Society **110** (2015), no. 3, 565–592.

[Kas15b] ———, *On the K-theory of subgroups of virtually connected Lie groups*, Algebr. Geom. Topol. **15** (2015), no. 6, 3467–3483.

[Kas16] ———, *On the K-theory of linear groups*, Annals of K-Theory **1** (2016), no. 4, 441–456.

[Kas17] ———, *The asymptotic dimension of quotients by finite groups*, Proc. Amer. Math. Soc. **145** (2017), no. 6, 2383–2389.

[KNR] D. Kasprowski, A. Nicas, and D. Rosenthal, *Regular finite decomposition complexity*, J. Topol. Anal., online ready, DOI:10.1142/S1793525319500286.

[KW] Daniel Kasprowski and Christoph Winges, *Shortening binary complexes and commutativity of K-theory with infinite products*, arXiv:1705.09116.

[KW19] ———, *Algebraic K-theory of stable ∞-categories via binary complexes*, Journal of Topology **12** (2019), no. 2, 442–462.

[LRRV17] Wolfgang Lück, Holger Reich, John Rognes, and Marco Varisco, *Algebraic K-theory of group rings and the cyclotomic trace map*, Adv. Math. **304** (2017), 930–1020.

[Roe03] John Roe, *Lectures on coarse geometry*, University Lecture Series, vol. 31, American Mathematical Society, Providence, RI, 2003.

[Ros04] David Rosenthal, *Splitting with continuous control in algebraic K-theory*, K-Theory **32** (2004), no. 2, 139–166.

[RTY14] Daniel A. Ramras, Romain Tessera, and Guoliang Yu, *Finite decomposition complexity and the integral Novikov conjecture for higher algebraic K-theory*, J. Reine Angew. Math. **694** (2014), 129–178.

[RV18] Holger Reich and Marco Varisco, *Algebraic K-theory, assembly maps, controlled algebra, and trace methods*, Space—time—matter, De Gruyter, Berlin, 2018, pp. 1–50.

[Sch06] Marco Schlichting, *Negative K-theory of derived categories*, Math. Z. **253** (2006), no. 1, 97–134.

[Sel60] Atle Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960, pp. 147–164.

[Weg12] Christian Wegner, *The K-theoretic Farrell-Jones conjecture for CAT(0)-groups*, Proc. Amer. Math. Soc. **140** (2012), no. 3, 779–793.

[Weg15] Christian Wegner, *The Farrell-Jones conjecture for virtually solvable groups.*, J. Topol. **8** (2015), no. 4, 975–1016.

[Win13] Christoph Winges, *A note on the L-theory of infinite product categories.*, Forum Math. **25** (2013), no. 4, 665–676.